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We propose a scalable procedure to generate entangled superpositions of motional coherent states and electronic states in N trapped ions. Beyond their fundamental interest, these states may be used in quantum information processing and for decoherence studies.

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In the last years large attention has being given to the possibility of producing superposition of states where a great number of particles would be involved. Beside their possible applications in quantum computation [1] and cryptography [2], they are an important tool in the experimental studies of decoherence [3]. In the microscopic world, a system can exist in a superposition of different quantum states, given rise to interference effects. On the other hand, these superpositions do not manifest themselves in the classical world [4]. This apparent paradox is solved if coherence is lost increasingly fast with the number of degrees of freedom of the system. Therefore, the study of the time scale in which decoherence occurs, as a function of the number of degrees of freedom of the system, is an important step for understanding the boundary between classical physics and quantum mechanics. The system of N trapped ions is a good laboratory for analyzing how decoherence appears when the system grows, since it is weakly affected by the environment [5]. In this paper, we discuss some proposals for generating vibronic superpositions of mesoscopic ion states in a fast and controllable way.

Let us consider N two-level ions of mass m, confined to move in the z direction in a Paul trap. They are cooled down to very low temperatures [6,7] and may perform small oscillations around their equilibrium positions, $z_{j0}, j = 1, 2...N$. We denote by $Z = \sum_{j=1,N} z_j/N$, the center of mass coordinate and we set the origin at its equilibrium position. All ions are simultaneously illuminated by two classical homogeneous Raman effective pulses $\vec{E}_I = \vec{E}_{0I} e^{i(\vec{q}_1\cdot\vec{r}-\omega_I t-\varphi_I)}$ and $\vec{E}_{II} = \vec{E}_{0II}\, e^{i(\vec{q}_2\cdot\vec{r}-\omega_{II} t-\varphi_{II})}$, with angular frequencies ω_I and ω_{II} and wave vectors $\vec{q}_1 = \vec{q}_2 = \vec{q}$, parallel to the z direction. The Raman pulses frequencies will be chosen to be quasi-resonant with a long-living electronic transition between two ionic hyperfine levels $|e_j\rangle$ and $|g_j\rangle$ $(j = 1, ...N)$, with energies $\hbar\omega_0$ and 0, respectively. The Raman laser relative phases are chosen to be the same $\varphi_I = \varphi_{II} = \varphi$. The total Hamiltonian of the system may be written, in the optical rotating wave approximation (RWA), as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}, \tag{1}$$

with

$$\hat{H}_0 = \hbar\omega_0 \sum_{j=1,N} |e_j\rangle\langle e_j| + \hbar\nu\hat{a}^\dagger\hat{a} + \sum_{\lambda=1,N-1} \hbar\nu_\lambda \hat{b}_\lambda^\dagger \hat{b}_\lambda\,, \tag{2}$$

and

$$\hat{H}_{\rm int} = \hbar\Omega \sum_{j=1,N} e^{i(q\hat{z}_j-\varphi)} |e_j\rangle\langle g_j| \left(e^{-i\omega_I t} + e^{-i\omega_{II} t}\right) + \text{H.c.} \tag{3}$$

The operators $\hat{a}$ and $\hat{b}_\lambda$ ($\hat{a}^\dagger$ and $\hat{b}_\lambda^\dagger$) are the annihilation (creation) operators associated with the center of mass mode of frequency ν and with the $N-1$ other vibrational modes of frequency ν_λ, respectively. For simplicity, we have assumed that the same Rabi frequency Ω (taken as real) is associated with both lasers.

We start by taking the frequencies ω_I and ω_{II} resonant with the center of mass vibronic transition in the k-th blue and k-th red sideband as

$$\omega_I = \omega_0 + k\nu \quad \text{and} \quad \omega_{II} = \omega_0 - k\nu\,. \tag{4}$$

For small k values, we may safely assume that only the center of mass motion will be excited, given that the next eigenfrequency is $\nu_r = \sqrt{3}\nu$, corresponding to the stretch mode. The following frequencies ($\geq \sqrt{29/5}\nu$)depend on the number of ions and have being calculated in Ref. [8]. Following the usual treatment for one single ion interacting with a laser field [9], we make the RWA with respect to the CM vibrational frequency and select the terms that oscillate with minimum frequency. For simplifying the notation, we define the "spin angular momentum states" $|\uparrow_j\rangle = e^{iqz_{j0}}|e_j\rangle$ and $|\downarrow_j\rangle = |g_j\rangle$. In the Lamb-Dicke limit the interaction Hamiltonian may be written, in the interaction picture, as

$$H_{\rm int} = \frac{2\Omega\hbar\eta^k}{k!}\hat{J}_{\rm T}(\hat{a}^k + \hat{a}^{\dagger k}) \tag{5}$$

where $\eta = q\sqrt{\hbar/2Nm\nu}$ is the Lamb-Dicke parameter, $\hat{J}_{\rm T}$ is an angular momentum-like operator defined as

$$\hat{J}_{\rm T} = \frac{i^k e^{-i\varphi}}{2} \sum_{j=1,N} \hat{\sigma}_{j+} + \text{H.c.} \tag{6}$$

Here $\widehat{\sigma}_{+j} = |\uparrow_j\rangle\langle\downarrow_j| = e^{iqz_{j0}}|e_j\rangle\langle g_j|$ is a flip operator associated with the electronic transition $|g_j\rangle \rightarrow |e_j\rangle$ in the ion j and $\widehat{\sigma}_{j-} = \widehat{\sigma}_{j+}^\dagger$. Without loss of generality we may set the phase $\varphi = k\pi/2$, so that $\hat{J}_{\rm T} = \hat{J}_x$ in the usual angular momentum operator convention for phases.

Similarly, by choosing a phase $\varphi = (k+1)\pi/2$, $\hat{J}_{\rm T} = \hat{J}_y$. We also may define the z component of the angular momentum by $\hat{J}_z = -i[\hat{J}_x, \hat{J}_y]$.

From Eq. 5, it is easy to show that the time evolution operator, in the interaction picture, at time t, is a sum of products of unitary operators on the motional states and projection operators on the ion internal states

$$\hat{U}_k(t) = \sum_{j,m} \hat{D}_k(m\alpha_k(t))|j,m\rangle_x\langle j,m| \tag{7}$$

where

$$\hat{D}_k(\alpha_k) = e^{\alpha_k \hat{a}^{k\dagger} - \alpha_k * \hat{a}^k}\,, \tag{8}$$

with $\alpha_k(t) = 2i\Omega t\eta^k/k!$. Also $|j,m\rangle_x$ are the simultaneous eigenvectors of the operators $\hat{J}_x$ and $\hat{J}^2 \equiv \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ associated with the eigenvalues $m = -j, -(j-1), ...j$ and $j(j+1)$, respectively. j varies from 0 (1/2) to $N/2$ by steps of 1, if N is even (odd).

The action of the time evolution operator $\hat{U}_k$ of Eq. (7) corresponds to unitary operations $\hat{D}_k(m\alpha_k)$ on the motional degrees of freedom conditioned to the value m of the x component of the "angular momentum" electronic state. When $k = 1$ that is, the excitation occurs in the first red and blue sidebands, $\hat{D}_1(\alpha_1)$ is the displacement operator which generates coherent states of the vibrational motion of the center of mass

$$|\alpha\rangle_{\rm coh} = e^{(\alpha\hat{a}^\dagger - \alpha*\hat{a})}|0\rangle = e^{-|\alpha|^2/2}\sum \frac{\alpha^n}{\sqrt{n!}}|n\rangle\,. \tag{9}$$

When $k = 2$, that is, the excitation occurs in the second red and blue sidebands, $\hat{D}_2(\alpha_2)$ is the squeezing operator that creates two photon coherent states [10]

$$|\alpha\rangle_{\rm sq} = e^{(\alpha\hat{a}^{2\dagger} - \alpha^*\hat{a}^2)}|0\rangle\,. \tag{10}$$

From now on, we consider only the case $k = 1$. Similar results will be valid for $k > 1$. If initially the ions are in the ground state, $|ggg...\rangle\otimes|0\rangle = |N/2, -N/2\rangle_z\otimes|0\rangle$, their state, after an interaction time t with the laser fields, will be given by

$$\sum_{m,m'} d^{N/2}_{m',m}(\pi/2)d^{N/2}_{m,-N/2}(-\pi/2)|N/2,m'\rangle_z \otimes |m\alpha\rangle_{\rm coh}, \tag{11}$$

where $\alpha = 2i\eta\Omega t$. Here, $d^j_{m',m}(\theta) =_z \langle j,m'|e^{-i\theta J_y}|j,m\rangle_z$ are the matrix elements of the rotation operator along the y axis in the $\{|j,m\rangle_z\}$ basis, eigenstates of $\hat{J}_z$ and $\hat{J}^2$ [11]. In Eq. (11), we have generated in a very simple way, and with a single Raman lasers pulse, a mesoscopic superposition of vibronic quantum states in N trapped ions. If we now measure the electronic state of the ions and find the totally excited state $|eee...\rangle$, we know that the motional state is given by

$$\frac{N!}{2^N}\sum_{m=-N/2,N/2} \frac{(-1)^{(N/2-m)}}{(N/2+m)!(N/2-m)!}|m\rangle_{\rm coh}\,. \tag{12}$$

This can be done by monitoring the fluorescence of a cyclic transition of the ions [12], where a dark event detects the totally excited state. If N is even, the motional state is a superposition of the vacuum and a series of even coherent states of amplitudes $m\alpha$, the probability for measuring them decreasing with $|m|$. If N is odd, the motional state is a superposition of two even coherent states of amplitudes $\pm N\alpha/2$. This is the superposition we have considered in Eq. (1). In Fig. 1, we show the Wigner function for the state given by Eq. (12), when $|\alpha| = 2$ and $N = 8$. The average amplitude of oscillation, $|N\alpha/2|^2$, is proportional both to N and τ^2. This state is an example of a strongly entangled mesoscopic state that is scalable, and may become a useful tool in decoherence studies.

If, after the preparation of the state given in Eq. (14), we measure the state $|eee...\rangle$, we obtain

$$\frac{|N\alpha/2\rangle_{\rm coh} + |-N\alpha/2\rangle_{\rm coh}}{\sqrt{2+2e^{-|N\alpha|^2/2}}}\,. \tag{15}$$

The state given in Eq. (15) is an even coherent state of the center of mass motion of an odd large number of ions. An odd coherent state is obtained if one measures the state $|ggg...\rangle$ instead of $|eee...\rangle$. Thus, we have a procedure that generates a superposition of two states of motion, vibrating out of phase, where a large number of particles may be involved. In principle, the number of ions in this state is limited mostly by decoherence and by the capacity of producing lasers with high homogeneity. For large N, the probability to produce these state superpositions by this procedure decreases as $1/2^N$. However, an efficient method to obtain even or odd coherent states with certainty, even for large N, may be found and will be described below.

As before, we start with the ground state and apply in succession the dispersive bichromatic interaction of Eq. (13) and the resonant bichromatic interaction to obtain the state given in Eq. (14). We then apply again the same dispersive bichromatic interaction during the same interval of time. The resulting state is

$$\frac{1}{2}\{|eee..\rangle\otimes(|N\alpha/2\rangle - |-N\alpha/2\rangle) \\ -|ggg...\rangle\otimes(|N\alpha/2\rangle + |-N\alpha/2\rangle)\} \tag{16}$$

Measuring the dark fluorescence of either the $N-$excited state or the $N-$ground state, we get the associated odd or even coherent state. In consequence, we showed how to generate efficiently mesoscopic superposition states for a large number of ions.

Similar results may be obtained for N even, if additional carrier $\pi/2$ pulses, with relative phases $\varphi = \pi/2$ with respect to the dispersive pulses, are applied. For example, if we want to prepare an even or odd coherent state with high probability, we should apply carrier

pulses just after the dispersive bichromatic interactions. This succession of pulses corresponds to the following time evolution operator

$$\hat{U} = e^{-iJ_y\pi/2} e^{-iJ_y^2\pi/2} e^{\hat{J}_x(\alpha\hat{a}^\dagger - \alpha^*\hat{a})} e^{-iJ_y\pi/2} e^{-iJ_y^2\pi/2}, \quad (17)$$

where $\alpha = -i\eta\Omega t$. The first and second pulses, as well as the fourth and the fifth, could be realized simultaneously, due to commutation properties, reducing the effective time of the process. The resulting state is given by

$$\begin{aligned}\hat{U}|ggg...\rangle = & \tfrac{1}{2}\left[(|N\alpha/2\rangle + |-N\alpha/2\rangle)\otimes|ggg...\rangle\right. \\ & \left. + i(|N\alpha/2\rangle - |-N\alpha/2\rangle)\otimes|eee...\rangle\right]. \quad (18)\end{aligned}$$

When $N = 2$, we do not need to apply the carrier pulses to obtain the even and odd coherent states. By applying the dispersive bichromatic interaction (with $\varphi = \pi/4$) on the ground state we generate the Bell state $\frac{1}{\sqrt{2}}(|ee\rangle + |gg\rangle)$ [13,16]. This state is also equal to the state

$$\frac{1}{\sqrt{2}}(|1,1\rangle_x + |1,-1\rangle_x). \quad (19)$$

We then apply the resonant bichromatic interaction (with $\varphi = \pi/2$) during a time τ to obtain the state

$$\frac{1}{\sqrt{2}}(|1,1\rangle_x|\alpha\rangle_{\rm coh} + |1,-1\rangle_x|-\alpha\rangle_{\rm coh}), \quad (20)$$

which can be written as

$$\begin{aligned}&\tfrac{1}{2}\left[(|ee\rangle + |gg\rangle)(|\alpha\rangle_{\rm coh} + |-\alpha\rangle_{\rm coh})\right. \\ &\left. + (e^{i\phi/2}|eg\rangle + e^{-i\phi/2}|ge\rangle)(|\alpha\rangle_{\rm coh} - |-\alpha\rangle_{\rm coh})\right] \quad (21)\end{aligned}$$

This state is an entangled state of the internal states of the ions with the two orthogonal even and odd coherent states of the center of mass motion. Direct measurement of the $|ee\rangle$ ($|gg\rangle$), by a dark fluorescence method will generate the even (odd) coherent state,

$$\frac{|\alpha\rangle \pm |-\alpha\rangle}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}}, \quad (22)$$

with probability $P = \frac{1\pm e^{-2|\alpha|^2}}{2}$.

We now consider the simultaneous application of four Raman lasers on N ions. We will consider that all lasers have the same amplitude and that the excitations are in the first sidebands. Their frequencies are chosen as

$$\begin{aligned}\omega_I = \omega_0 + k\nu\,, \qquad \omega_{II} = \omega_0 - k\nu\,, \\ \omega_{III} = \omega_0 + k\nu_r\,, \qquad \omega_{IV} = \omega_0 - k\nu_r\,, \quad (23)\end{aligned}$$

where $\nu_r = \sqrt{3}\nu$. We take the phases of all effective Raman lasers pulses the same and equal to $\varphi = k\pi/2$. Using a similar reasoning, as the one used in the resonant bichromatic excitation of the CM mode only, it is easy to see that the time evolution operator is now given by

$$\hat{U}(t) = \sum_{j,m} \hat{D}_k(m\alpha_{CM}, m\alpha_r)|j,m\rangle_x\langle j,m|. \quad (24)$$

Here,

$$\hat{D}_k(\alpha_{CM},\alpha_r) = e^{(\alpha_{CM}\hat{a}^{k\dagger} - \alpha_{CM}^*\hat{a}^k)} e^{(\alpha_r\hat{b}^{k\dagger} - \alpha_r^*\hat{b}^k)}, \quad (25)$$

where $\alpha_{CM} = 2i\Omega t\eta^k/k!$, $\alpha_r = \sqrt{3}\alpha_{CM}$, b and $b^\dagger$ are the annihilation and creation operator for the stretch mode.

Consider for simplicity the case $k = 1$ with $N = 2$. Similarly to the discussion of the bichromatic interaction, if the initial state is the ground state, the application of the four Raman lasers during a time t, followed by the detection of state $|ee\rangle$, generates the entangled state

$$\mathcal{N}\left(|\alpha_{CM}\alpha_r,\rangle_{\rm coh} + |-\alpha_{CM},-\alpha_r\rangle_{\rm coh} - 2|0,0\rangle\right), \quad (26)$$

where $\mathcal{N}$ is a normalization constant, with probability

$$P = \frac{6 - 8e^{-(|\alpha_{CM}|^2+|\alpha_r|^2)/2} + 2e^{-2(|\alpha_{CM}|^2+|\alpha_r|^2)}}{16}. \quad (27)$$

Analogously, if the initial state is the Bell state $|ee\rangle + |gg\rangle$, the application of the four Raman lasers during a time τ, followed by the detection of state $|ee\rangle$, generates the state

$$\frac{|\alpha_{CM}\alpha_r,\rangle_{\rm coh} + |-\alpha_{CM},-\alpha_r\rangle_{\rm coh}}{\sqrt{2(1+e^{-2|\alpha_{CM}|^2-2|\alpha_r|^2})}} \quad (28)$$

with probability 1/2.

In conclusion, we have presented a procedure to generate several kinds of mesoscopic superpositions of states involving N ions. Coherent and odd coherent states may be generated rapidly, through resonant interactions, with a probability that decreases exponentially fast $(1/2^N)$. High probability generation (1/2), may be obtained if we use also dispersive bichromatic interactions as an intermediate step. Then, the number of ions involved will be limited mostly by the scale of time in which decoherence occurs. We believe that the proposals presented in this letter are ready to be implemented in the laboratory. They should help to build larger mesoscopic quantum superpositions in trapped ions, to study at large scale decoherence processes and for applications in quantum information processing.

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